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Null Field Approach to Scalar Diffraction. III. Inverse Methods

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NULL FIELD APPROACH TO SCALAR DIFFRACTION
III. INVERSE METHODS

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We show how our null field methods might be adapted to provide a sharp numerical test for the radius at which a series expansion of a scattered field starts to diverge. On the basis of our spherical and cylindrical physical optics approximations we develop an inversion procedure, similar to conventional procedures based on planar physical optics and like them needing scattering data over a wide range of frequencies, suitable for totally reflecting bodies. We introduce another method, also based on spherical and circular physical optics, whereby the shapes of certain bodies of revolution and cylindrical bodies can be reconstructed from scattered fields observed for only two closely spaced frequencies. We present computational examples which confirm the potential usefulness of the latter method.

1. Introduction

This is the final paper in a series of three in which a computationally orientated approach to diffraction theory is developed from the optical extinction theorem (extended boundary condition). In the first two papers (Bates & Wall 1977*a*, *b*), which are henceforth referred to as (I) and (II) respectively, we examined direct scattering methods. Here we discuss inverse scattering.

The general inverse scattering problem is posed as: determine the shape and constitution of a scattering body, given the incident field and the scattered far field. De Goede (1973) shows that the extinction theorem can be inverted to give an integral equation for the material constituents of an inhomogeneous medium in terms of the field existing at the boundary of the medium. Unfortunately, the kernel of the integral involves a propagator (Green's function) which itself depends upon the material constituents, so that the problem cannot be said to be reduced to a form whereby the solution can be computed; nevertheless, this is a comparatively new approach which, hopefully, will be developed further. The established inversion technique with the widest application is Gel'fand and Levitan's method (cf. Newton 1966) which has been most highly developed by Kay & Moses (1961) and Wadati & Kamijo (1974); a method of wider potential applicability has recently been suggested (Bates 1975*b*).

In most situations of physical interest a fair amount of information concerning the general shape and/or size and/or material constitution of the scattering body is available *a priori*. Because of this, many specialized inverse scattering problems have been posed (cf. Colin 1972).

We consider only totally reflecting bodies here. Our main intention is to make clear both the power and the limitations of our methods. Accordingly, we restrict our detailed analysis to sound-soft bodies. The corresponding analysis for sound-hard bodies is only different in detail, and we do not take the space to examine it explicitly.

In § 2 we gather from (I) and (II) the formulae needed here. Since it is the shape of a body which has to be determined from observation of its scattered field, it seems pointless to employ coordinate systems especially suitable for bodies of particular aspect ratios. Consequently, we only invoke the spherical null field method for bodies of arbitrary shape, and the circular null field method, for cylindrical bodies. Section 3 outlines the relevance of the null field method to the exact approach to inverse scattering based on analytical continuation (see Weston, in Colin 1972). Section 4 is concerned with an alternative to the usual inversion procedures based on planar physical optics (cf. Bojarski, in Colin 1972). Like those who have gone before us, we need to know the scattered field over a wide range of frequencies; but our technique seems to apply to a broader class of bodies. The main contribution of this paper is introduced in § 5, where we show that the shapes of certain bodies can be reconstructed from scattered fields observed at two closely spaced frequencies. The computational examples presented in § 6 confirm that useful results can be obtained in situations of physical interest. In § 7 we attempt to assess the significance of our work in relation to current research into the inverse scattering problem.

2. Preliminaries

Figure 1 shows the surface S of a totally reflecting body of arbitrary shape embedded in the three-dimensional space Υ, which is partitioned into Υ_- and Υ_+, the regions inside and outside S respectively. A point O within Υ_- is taken as origin for a spherical polar coordinate system. We denote arbitrary points in Υ and on S by P, with coordinates (r, θ, ϕ), and P', with coordinates

(r', θ', ϕ'), respectively. We denote the points on S closest to, and furthest from, O by $P'_{\min}$ and $P'_{\max}$ respectively. The radial coordinates of $P'_{\min}$ and $P'_{\max}$ are $r'_{\min}$ and $r'_{\max}$ respectively. We denote by Υ_{null} the parts of Υ_- within which $r < r'_{\min}$. We denote by Υ_{++} the parts of Υ_+ within which $r > r'_{\max}$. The remaining parts of Υ_- and Υ_+ are Υ_{-+} and Υ_{+-} respectively, as is indicated in figure 1. Extensions of this notation are defined in § 2 of (I) and § 2 of (II).

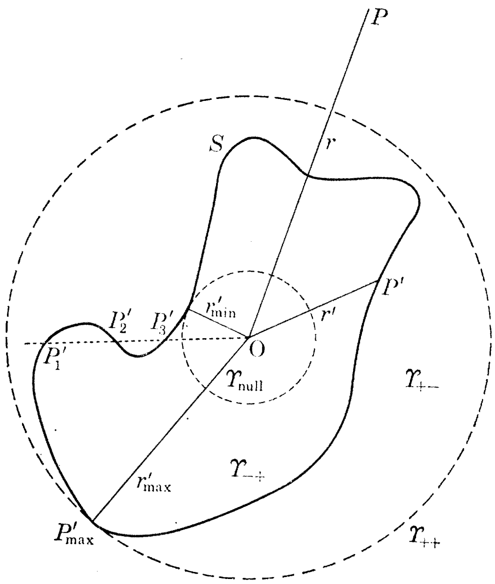

FIGURE 1. Totally reflecting scattering body of arbitrary shape.

In conformity with § 2 (*b*) of (II) we introduce the spherical physical optics 'illuminated' and 'shadowed' parts of S, called S_+ and S_- respectively. These are carefully defined in (II). Here it is sufficient to remark that $P' \in S_+$ if and only if the extension of its radial coordinate from O does not again intersect S. Refer to the points P'_1, P'_2 and P'_3 lying on the straight, dashed line shown in figure 1. We see that $P'_1 \in S_+$ whereas $P'_2, P'_3 \in S_-$. We also need to partition S in another way, when considering the behaviour of fields in Υ_{-+} and Υ_{+-}. We define

$$S \sim S^-(r) \cup S^+(r), \quad P' \in \begin{matrix} S^-(r) & (r' > r) \\ S^+(r) & (r' \leqslant r). \end{matrix} \Bigg\} \tag{2.1}$$

Note that $S^-(r)$ is empty when $r > r'_{\max}$, and $S^+(r)$ is empty when $r < r'_{\min}$.

We now recall (2.5), (2.8) and (2.10) all of (I) and we abstract certain formulas from § 2 (*a*) of (I). The sources of the monochromatic field – denoted by $\psi_0 = \psi_0(r, \theta, \phi, k)$ – incident upon the body are confined to Υ_{++}. So, we can write ψ_0 as

$$\psi_0 = \sum_{l=0}^{\infty} \sum_{j=-l}^{l} c_{j,l} a_{j,l}(k) j_l(kr) P_l^j(\cos\theta) \exp(\mathrm{i}j\phi), \quad 0 \leqslant r \leqslant r'_{\max}, \ (0 \leqslant \phi < 2\pi, \ 0 \leqslant \theta \leqslant \pi), \tag{2.2}$$

where the $j_l(\cdot)$ are spherical Bessel functions of the first kind. The $a_{j,l} = a_{j,l}(k)$ are expansion coefficients which determine the precise form of ψ_0, and k is the wave number. The time factor $\exp(\mathrm{i}\omega t)$ is suppressed. The normalization constants $c_{j,l}$ are given by

$$c_{j,l} = -\mathrm{i}k \frac{(l-j)!}{(l+j)!} \frac{(2l+1)}{4\pi}. \tag{2.3}$$

The scattered field $\psi = \psi(r, \theta, \phi, k)$ can be written as

$$\psi = \sum_{l=0}^{\infty} \sum_{j=-l}^{l} c_{j,l}[B^{-}_{j,l}(r,k)\, j_l(kr) + B^{+}_{j,l}(r,k)\, h^{(2)}_l(kr)]\, P^j_l(\cos\theta) \exp(\mathrm{i}j\phi) \quad (P \in \Upsilon), \tag{2.4}$$

where, for $l \in \{0 \to \infty\}$ and $j \in \{-l \to l\}$,

$$B^{\beta}_{j,l}(r,k) = \int_{S^{\beta}(r)}\!\!\int f(\tau_1, \tau_2)\, w^{\beta}_l(kr')\, P^j_l(\cos\theta') \exp(-\mathrm{i}j\phi')\, \mathrm{d}s, \tag{2.5}$$

where β denotes either $-$ or $+$, and $h^{(2)}_l(\cdot)$ denotes the spherical Hankel function of the second kind, and

$$w^{+} = j \quad \text{and} \quad w^{-} = h^{(2)} \tag{2.6}$$

and $f(\tau_1, \tau_2)$ is the density of reradiating sources induced in the surface (in which τ_1 and τ_2 are convenient, orthogonal, parametric coordinates) of the sound-soft body. Note that ψ, as given by (2.4), satisfies the radiation condition at infinity because $S^{-}(r)$ is empty for $r > r'_{\max}$. In conformity with notation introduced in (I) and (II) we write

$$\left.\begin{aligned} B^{-}_{j,l}(r,k) &= b^{-}_{j,l}(k) \quad r < r'_{\min}, \\ B^{+}_{j,l}(r,k) &= b^{+}_{j,l}(k) \quad r > r'_{\max}. \end{aligned}\right\} \tag{2.7}$$

We can compute $f = f(\tau_1, \tau_2)$ by solving the null field equations:

$$b^{-}_{j,l}(k) + a_{j,l} = 0 \quad (l \in \{0 \to \infty\}, j \in \{-l \to l\}). \tag{2.8}$$

For the approximate approaches developed in §§ 4, 5 we need the form of the spherical physical optics surface source density when the incident field is characterized by

$$a_{j,l}(k) = 0 \quad (l > 0) \tag{2.9}$$

the physical implications of which are discussed in the Appendix. The normalization

$$a_{0,0}(l) = -4\pi\mathrm{i} \tag{2.10}$$

in convenient. It follows from § 2 (*b*) of (II) that the spherical physical optics surface source density is

$$\begin{aligned} \tilde{f}(\theta', \phi') &= 0, \quad P' \in S_{-} \\ &= \frac{kr' \sin(\theta')}{\Delta(\theta', \phi')} \exp(\mathrm{i}kr'), \quad P' \in S_{+}, \end{aligned} \tag{2.11}$$

where $\Delta(\theta', \phi') = \mathrm{d}s/\mathrm{d}\theta'\,\mathrm{d}\phi'$. Note that we have used the formulae

$$P^0_0(\cos\theta') = 1 \quad \text{and} \quad h^{(2)}_0(kr') = (\mathrm{i}/kr') \exp(-\mathrm{i}kr'). \tag{2.12}$$

We recall from § 2 (*b*) of (II) that the coordinates ϕ' and θ' span S_{+} single-valuedly and continuously throughout the ranges $[0, 2\pi]$ and $[0, \pi]$ respectively. So, if we replace $f(\cdot)$ in (2.5) by $\tilde{f}(\cdot)$, and it we note (2.1) and (2.7), see that

$$k \int_0^{\pi} \int_0^{2\pi} r' \exp(\mathrm{i}kr')\, j_l(kr')\, P^j_l(\cos\theta') \exp(-\mathrm{i}j\phi') \sin(\theta')\, \mathrm{d}\phi'\, \mathrm{d}\theta' \approx b^{+}_{j,l}(k)$$
$$(l \in \{0 \to \infty\}, j \in \{-l \to l\}) \tag{2.13}$$

on account of (2.6) and (2.11). We use an 'approximately equals' sign in (2.13) because we have invoked the physical optics surface source density rather than the exact surface source density, but this is the only approximation implicit in (2.13). We define

$$E(\theta, \phi, k) = \sum_{l=0}^{\infty} (2l+1)\, \mathrm{i}^l \sum_{j=-l}^{l} \frac{(l-j)!}{(l+j)!} b^{+}_{j,l}(k)\, P^j_l(\cos\theta) \exp(\mathrm{i}j\phi), \tag{2.14}$$

which, when combined with (2.13), leads to

$$k\int_0^{\pi}\int_0^{2\pi} r'\exp\left(\mathrm{i}kr'[1+\cos(\Theta)]\right)\sin(\theta')\,\mathrm{d}\phi'\,\mathrm{d}\theta' \approx E(\theta,\phi,k) \tag{2.15}$$

because

$$\sum_{l=0}^{\infty}(2l+1)\,\mathrm{i}^l P_l^0(\cos\Theta)\,j_l(kr') = \exp(\mathrm{i}kr'\cos\Theta) \tag{2.16}$$

and

$$P_l^0(\cos\Theta) = \sum_{j=-l}^{l}\frac{(l-j)!}{(l+j)!}P_l^j(\cos\theta)\,P_l^j(\cos\theta')\exp(\mathrm{i}j[\phi-\phi']), \tag{2.17}$$

where (cf. Abramowitz & Stegun 1968, Chs 8 and 10)

$$\cos(\Theta) = \cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\phi-\phi'). \tag{2.18}$$

(a) *Cylindrical body*

When neither the fields nor the cross section of the body exhibit any variation in the direction perpendicular to the plane of figure 1 then S can be replaced by C, which is the cross section in a particular plane denoted by Ω. Cylindrical polar coordinates are used to identify P and P', i.e. (ρ,ϕ) and (ρ',ϕ') respectively. The previous notation is modified accordingly.

We now list formulae needed later. It is, however, worth referring to §2(*c*) of (II). The incident field is written as

$$\psi_0 = (-\tfrac{1}{4}\mathrm{i})\sum_{m=0}^{\infty}\epsilon_m[a_m^{\mathrm{e}}(k)\cos(m\phi) + a_m^{\mathrm{o}}(k)\sin(m\phi)]\,J_m(k\rho) \quad (0 \leqslant \rho \leqslant \rho'_{\max},\ 0 \leqslant \phi \leqslant 2\pi), \tag{2.19}$$

where the sources of ψ_0 are confined to parts of Ω for which $\rho \leqslant \rho'_{\max}$. The Neumann factor $\epsilon_m = 1$ for $m = 0$, but $\epsilon_m = 2$ for $m > 0$. We write the scattered field in the form

$$\psi = (-\tfrac{1}{4}\mathrm{i})\sum_{m=0}^{\infty}\epsilon_m[b_m^{+\mathrm{e}}(k)\cos(m\phi) + b_m^{+\mathrm{o}}(k)\sin(m\phi)]\,H_m^{(2)}(k\rho), \quad P\in\Omega_{++}, \tag{2.20}$$

where

$$b_m^{+\times}(k) = \int_C F(C)\,J_m(k\rho')\,\Psi^{\times}(m\phi')\,\mathrm{d}C \quad (m\in\{0\to\infty\}), \tag{2.21}$$

where the superscript $\times$ denotes either e or o, and Ψ^{e} represents cos, and Ψ^{o} represents sin, and $F(C)$ is the surface source density.

When the incident field is characterized by

$$a_m^{\times} = 0 \quad (m > 0) \quad \text{and} \quad a_0^{\mathrm{o}} = 0 \tag{2.22}$$

and the normalization

$$a_0^{\mathrm{e}} = -(8\pi\mathrm{i})^{\frac{1}{2}} \tag{2.23}$$

is made, the circular physical optics surface source density becomes

$$\tilde{F}(\phi') = 0, \quad P\in C_-$$
$$\approx \frac{\mathrm{d}\phi'}{\mathrm{d}C}(k\rho')^{\frac{1}{2}}\exp(\mathrm{i}k\rho'), \quad P\in C_+, \tag{2.24}$$

where the 'approximately equals' sign is used because there is no exact formula of the same kind as the second one in (2.12) for $H_0^{(2)}(k\rho)$. However, if $k\rho'_{\min} > 2\pi$, the formula

$$H_0^{(2)}(k\rho') = (\mathrm{i}2/\pi k\rho')^{\frac{1}{2}}\exp(-\mathrm{i}k\rho') \tag{2.25}$$

is less than 2 % in error. The formula corresponding to (2.13) is

$$k^{\frac{1}{2}}\int_0^{2\pi}(\rho')^{\frac{1}{2}}\exp(\mathrm{i}k\rho')\,J_m(k\rho')\,\Psi^{\times}(m\phi')\,\mathrm{d}\phi' \approx b_m^{+\times}(k) \quad (m\in\{0\to\infty\}). \tag{2.26}$$

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We define
$$E(\phi,k) = \sum_{m=0}^{\infty} \epsilon_m \mathrm{i}^m [b_m^{+\mathrm{e}}(k)\cos(m\phi) + b_m^{+\mathrm{o}}(k)\sin(m\phi)], \tag{2.27}$$

which, when combined with (2.26), gives

$$k^{\frac{1}{2}} \int_0^{2\pi} (\rho')^{\frac{1}{2}} \exp(\mathrm{i}k\rho'[1+\cos(\phi-\phi')])\,\mathrm{d}\phi' \approx E(\phi,k) \tag{2.28}$$

because
$$\sum_{m=0}^{\infty} \epsilon_m \mathrm{i}^m \cos(m[\phi-\phi'])\,J_m(k\rho') = \exp(\mathrm{i}k\rho'\cos(\phi-\phi')). \tag{2.29}$$

(b) *Inverse scattering problem*

Because of (2.1) and the sentence following it, and because of (2.4) through (2.7), it follows that

$$\psi = \sum_{l=0}^{\infty} \sum_{j=-l}^{l} c_{j,l} b_{j,l}(k)\, h_l^{(2)}(kr)\, P_l^j(\cos\theta) \exp(\mathrm{i}j\phi), \quad P \in \Upsilon_{++}. \tag{2.30}$$

The equivalent formula for cylindrical bodies is (2.20). The available data for the inverse scattering problem are the scattered far field and the incident field throughout $\Upsilon_- \cup \Upsilon_{+-}$ (it may also be known within a large part of Υ_{++}, but this is strictly unnecessary). The incident field is characterized by the complete set of the $a_{j,l}(k)$, or the $a_m(k)$ for cylindrical bodies, or as many of them that have magnitudes exceeding a threshold set by the specified error permitted in the final solution to the problem. In the far field, the spherical Hankel functions appearing in (2.30) can, by definition, be replaced by the leading terms in their asymptotic expansions (cf. Abramowitz & Stegun 1968, Ch. 10). It follows that

$$\psi = -\frac{\exp(-\mathrm{i}kr)}{kr} \sum_{l=0}^{\infty} \sum_{j=-l}^{l} (\mathrm{i})^{l+1} c_{j,l} b_{j,l}^{+}(k)\, P_l^j(\cos\theta) \exp(\mathrm{i}j\phi), \quad P \in \Upsilon_{\mathrm{far}}, \tag{2.31}$$

where Υ_{far} is the part of Υ_{++} far enough away from the body to be in its far scattered field. Given ψ in the far field, for a particular r and for all ϕ and θ in the ranges $[0, 2\pi]$ and $[0, \pi]$ respectively, the complete set of $b_{j,l}(k)$ (or as many of them that have magntiudes exceeding an appropriate threshold) can be immediately obtained on account of the orthogonality of the functions $P_l^j(\cos\theta)\exp(\mathrm{i}j\phi)$. So, inspection of (2.30) indicates that, using the available data, ψ can be immediately computed anywhere within Υ_{++}. The problem is to reconstruct S.

Reference to (2.14) confirms that the available information concerning the scattered field is contained in $E(\theta,\phi,k)$. For cylindrical bodies the equivalent quantity is $E(\phi,k)$.

To recapitulate; we can pose the inverse scattering problem as: find S, given the $a_{j,l}(k)$ and the $b_{j,l}^{+}(k)$, or equivalently, given $E(\theta,\phi,k)$. For cylindrical bodies the problem is: find C, given the $a_m^{\times}(k)$ and the $b_m^{+\times}(k)$, or equivalently, given $E(\phi,k)$.

3. Formally exact approach

The uniqueness of analytical continuation ensures that (cf. Bates 1975 *a*)

$$\psi = \sum_{l=0}^{\infty} \sum_{j=-l}^{l} c_{j,l} b_{j,l}^{+}(k)\, h_l^{(2)}(kr)\, P_l^j(\cos\theta) \exp(\mathrm{i}j\phi), \quad P \in \bar{\Upsilon}_{+} \tag{3.1}$$

where $\overline{\Upsilon}_+$ is the part of Υ throughout which the right-hand side (r.h.s.) of (3.1) is uniformly convergent. It follows necessarily from (2.4) through (2.7) that

$$\overline{\Upsilon}_+ \supset \Upsilon_{++}. \tag{3.2}$$

When the scattering body and the incident field are such that $\overline{\Upsilon}_+ \supset \Upsilon_{+-}$ then the inverse scattering problem can be solved exactly, straightforwardly. The standard boundary condition for sound-soft bodies is

$$\psi + \psi_0 = 0, \quad P' \in S. \tag{3.3}$$

Since ψ_0 and the $b_{j,l}(k)$ are given (refer to § 2 (b)), r.h.s. (3.1) can be computed. It follows that we can easily find, by computation, the points $P'' \in \Upsilon$ where $\psi + \psi_0$ vanishes. Ordinary interference can cause the total field to vanish at points, along lines and even along surfaces none of which coincide with S. So, the points P'' must be found for sufficient wave numbers to ensure that the true surface is mapped out (only those P'' that reappear for all wave numbers are accepted as belonging to S). This approach seems to have been initiated by Weston & Boerner (1969); some of its computational consequences have been investigated by Imbriale & Mittra (1970).

Suppose ψ is given, or observed, for all θ and ϕ at $r = r_0$, where $r_0 > r'_{\max}$. An infinity of source distributions can be constructed to give rise to ψ for $r > r_0$ (cf. Sleeman 1973). However, Millar (1973, note also his previous work which he references in this paper) has shown that there is a unique closed curve in Ω (called the hull of the singularities, which is necessarily included in Ω_-) within which it is impossible to construct source distributions which could give rise to ψ. Millar has also shown that any particular series representation of ψ, of the form of (2.20), has unique singularities, because the series has a definite radius of convergence, i.e. uniform convergence occurs only if r is greater than this radius. So, the aforementioned infinity of source distributions must lie between the circle of radius r_0 and the hull of the singularities.

The hull of the singularities is found by successive applications of the addition theorem for Bessel functions, in order to construct series similar to (2.20) but for different coordinate origins (cf. Bates 1975a). Colton (1971) carries through a similar analysis for axially symmetric solutions to the Helmholtz equation. There seems to be no good reason for doubting that the field scattered from a body of arbitrary shape has a unique hull of singularities, which must be included in Υ_-. When this hull is included in Υ_{null} it is clear that r.h.s. (3.1) can be used to represent ψ in (3.3) for all $P' \in S$. When the hull intersects Υ_{-+} then r.h.s. (3.1) is not uniformly convergent throughout Υ_{-+} and some analytic continuation procedure (such as the employment of addition theorems) must be devised to solve (3.3) for all $P' \in S$.

The scattered field must be well behaved throughout Υ_{+-}. Consequently, the addition theorems for spherical wave functions can be invoked to continue r.h.s. (3.1) uniquely throughout Υ_{+-}, in much the same way as these theorems are employed in § 3 of (II), as Weston, Bowman & Ar (1968), Weston & Boerner (1969) and Imbriale & Mittra (1970) have investigated in detail. Ahluwalia & Boerner (1973, 1974) and Yerokhin & Kocherzhevskiy (1975) have extended the method to those sorts of penetrable bodies that can be usefully characterized by surfaces impedances.

Multiple use of addition theorems is time-consuming computationally, and care is needed to prevent errors accumulating. Also, one is trying to discover the shape of the body, so that it is by no means obvious which is the best position for the new coordinate origin when one is making a particular application of an addition theorem. Consequently, there are severe difficulties associated with analytical continuation methods, and these difficulties are accentuated by the usual problems with numerical stability (Cabayan, Murphy & Pavlasek 1973).

Analytical continuation methods would be easier to use if a sharp test could be devised for estimating the minimum value of r for which r.h.s. (3.1) is uniformly convergent. Inspection of (2.4) through (2.7) reveals that

$$\psi = \sum_{l=0}^{\infty}\sum_{j=-l}^{l} c_{j,l}[B^{-}_{j,l}(r,k)\,j_l(kr) - B^{+-}_{j,l}(r,k)\,h^{(2)}_l(kr) + b^{+}_{j,l}(k)\,h^{(2)}_l(kr)]\,P^j_l(\cos\theta)\exp(\mathrm{i}j\phi), \quad P \in \Upsilon, \tag{3.4}$$

where, for $l \in \{0 \rightarrow \infty\}$ and $j \in \{-l \rightarrow l\}$,

$$B^{+-}_{j,l}(r,k) = \int_{S^-(r)}\!\!\int f(\tau_1,\tau_2)\,j_l(kr')\,P^j_l(\cos\theta')\exp(-\mathrm{i}j\phi')\,\mathrm{d}s. \tag{3.5}$$

It follows necessarily from (3.1) that

$$\sum_{l=0}^{\infty}\sum_{j=-l}^{l} c_{j,l}\beta_{j,l}(r,k)\,P^j_l(\cos\theta)\exp(\mathrm{i}j\phi) = 0, \quad P \in \Upsilon_+, \tag{3.6}$$

where, for $l \in \{0 \rightarrow \infty\}$ and $j \in \{-l \rightarrow l\}$

$$\beta_{j,l}(r,k) = B^{-}_{j,l}(r,k)\,j_l(kr) - B^{+-}_{j,l}(r,k)\,h^{(2)}_l(kr). \tag{3.7}$$

The first value of r which will be found to satisfy (3.3) is $r'_{\max}$. Consider a particular value of r, say r_{p}, less than $r'_{\max}$. If all the points on S, for which $r' > r_{\mathrm{p}}$, are found from (3.3) then $S^-(r_{\mathrm{p}})$ is known, which means that $f(\tau_1,\tau_2)$ can be computed for all $P' \in S^-(r_{\mathrm{p}})$ by using (2.8) or (I). Reference to (2.5), (3.5) and (3.7) of this paper then confirms that $\beta_{j,l}(r_{\mathrm{p}},k)$ can be calculated for $l \in \{0 \rightarrow \infty\}$ and $j \in \{-l \rightarrow l\}$. For each $r = r_{\mathrm{p}}$, l.h.s. (3.6) can be computed. If there is found to be a value of r, which we denote by r_{crit}, for which

$$|\text{l.h.s. (3.6)}| > \text{threshold}, \quad r < r_{\mathrm{crit}}, \tag{3.8}$$

where the threshold is related to computational round-off errors and to the quality of the data, then we can assume that r.h.s. (3.1) is not uniformly convergent for $r < r_{\mathrm{crit}}$.

Similar reasoning to that developed in the previous paragraph has been previously presented for cylindrical bodies (Bates 1970). In this earlier analysis we suggested that analytical continuation would allow us to recover the whole of S, without having to use addition theorems. This is sound theoretically because the non-converging part of r.h.s. (3.1) is exactly cancelled by the non-converging part of l.h.s. (3.6), for all $r < r_{\mathrm{crit}}$. But we have never found a computationally satisfactory way of taking advantage of this, which is not surprising in the light of the results of Cabayan *et al.* (1973). However, we feel that the method for testing for r_{crit} described in the previous paragraph is computationally viable, because l.h.s. (3.6) is necessarily zero for $r > r_{\mathrm{crit}}$.

4. Approximate approach – all frequences

The positions of scattering bodies in space can be determined with useful accuracy in many sorts of situation by conventional radar and sonar techniques. The precision of the position measurement increases as the bandwidth of the transmissions is increased. Sophisticated systems have been developed for estimating the shapes, as well as the positions (and the velocities of moving bodies), of the bodies (cf. Bates 1969). Several estimation procedures involve various Fourier transformations of the scattered field, which is assumed to be close to that predicted by planar physical optics (cf. Bates 1969; Lewis 1969). Theoretically, the scattered field must be known for all frequencies, or wave numbers.

We present here an alternative inversion technique, for which we also require the complete scattered field at all frequencies. We base our procedure on spherical physical optics, which like planar physical optics becomes increasingly inappropriate as the wave number increases beyond a certain limit, corresponding roughly to where the largest linear dimension of the body equals the wavelength. However, as follows from the analysis developed in § 2 (*b*) of (II), we can claim that, when (2.9) of this paper applies, the form of the physical optics surface source density used by us is in general more accurate than the forms employed in previously reported inversion methods.

Multiplying (2.13) by $(2/\pi k^3)^{\frac{1}{2}}$ and integrating with respect to k from 0 to ∞ gives (cf. Watson 1966, § 13.42)

$$\mathrm{i}^l \int_0^{\pi}\int_0^{2\pi} (r')^{\frac{1}{2}} P_l^j(\cos\theta') \exp(-\mathrm{i}j\phi') \sin(\theta')\, \mathrm{d}\phi'\, \mathrm{d}\theta' \approx (2/\mathrm{i}\pi)^{\frac{1}{2}}(l+\tfrac{1}{2}) \int_0^{\infty} k^{-\frac{3}{2}} b_{j,l}^{+}(k)\, \mathrm{d}k \quad (l\in\{0\to\infty\}, j\{-l\to l\}). \tag{4.1}$$

Examination of l.h.s. (2.13), in the limit as $k\to 0$, indicates that r.h.s. (4.1) exists. Since r' is a single-valued function of θ' and ϕ' over S_+, we see that (4.1) leads immediately to

$$\{r'(\theta',\phi')\}^{\frac{1}{2}} \approx (32\pi^2)^{-\frac{1}{2}} \sum_{l=0}^{\infty} (2l+1)^2 (-\mathrm{i})^{l+\frac{1}{2}} \sum_{j=-l}^{l} \frac{(l-j)!}{(l+j)!} P_l^j(\cos\theta') \times \exp(\mathrm{i}j\phi') \int_0^{\infty} k^{-\frac{3}{2}} b_{j,l}^{+}(k)\, \mathrm{d}k \tag{4.2}$$

because

$$\frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)\sum_{j=-l}^{l}\frac{(l-j)!}{(l+j)!}P_l^j(\cos\theta)\,P_l^j(\cos\theta')\exp\{\mathrm{i}j(\phi-\phi')\} = \frac{\delta(\phi-\phi')\,\delta(\theta-\theta')}{\sin(\theta)}, \tag{4.3}$$

where $\delta(\cdot)$ denotes the Dirac delta function.

We obtain an estimate of the shape of S_+ from (4.2).

It is worth noting that (4.1) and (4.2) emphasize the necessity of defining physical optics surface source densities over parts of S which can be described single-valuedly by convenient coordinate systems. If r' were not necessarily a single-valued function of θ' and ϕ', we could not necessarily identify r.h.s. (4.2) with a single value of $(r')^{\frac{1}{2}}$.

(*a*) *Cylindrical body*

Integrating (2.26) with respect to k from 0 to ∞ gives (cf. Abramowitz & Stegun 1968, formula 11.4.12)

$$\mathrm{i}^m \int_0^{2\pi} (\rho')^{\frac{1}{4}}\, \Psi^{\times}(m\phi') \approx f_m \int_0^{\infty} k^{-\frac{5}{4}} b_m^{+\times}(k)\, \mathrm{d}k \quad (m\in\{0\to\infty\}), \tag{4.4}$$

where

$$f_m = \frac{2^{\frac{1}{4}}\exp(-\mathrm{i}\pi/8)\,\Gamma(\frac{1}{2})\,\Gamma(m+\frac{3}{4})}{\Gamma(m+\frac{1}{4})\,\Gamma(\frac{1}{4})} \tag{4.5}$$

and $\Gamma(\cdot)$ denotes the gamma function. As $\rho' = \rho'(\phi')$ is single-valued over C_+, (4.4) leads immediately to

$$(\rho'(\phi'))^{\frac{1}{4}} \approx \frac{1}{2\pi}\sum_{m=0}^{\infty}\epsilon_m(\mathrm{i})^{-m} f_m \int_0^{\infty} k^{-\frac{5}{4}}[b_m^{+\mathrm{e}}(k)\cos(m\phi') + b_m^{+\mathrm{o}}(k)\sin(m\phi')]\,\mathrm{d}k \tag{4.6}$$

because

$$\frac{1}{2\pi}\sum_{m=0}^{\infty}\epsilon_m\cos\{m(\phi-\phi')\} = \delta(\phi-\phi'). \tag{4.7}$$

We obtain an estimate of the shape of C_+ from (4.6).

(b) *Discussion*

The formulae (4.2) and (4.6) can be thought of as a development of the approach initiated by Freedman (1963) for sonar and by Kennaugh & Moffatt (1965) for radar. The chief weakness of our formulae from a practical point of view is that they require the scattered field to be observed over an indefinitely large range of frequencies. A significant relaxation of this requirement should be obtained before any extensive computational investigation is embarked upon. Young's (1976) impressive practical implementation of Kennaugh & Moffatt's approach shows that it is worth attempting to find appropriate modifications to the analysis presented here.

5. Approximate approach – two frequencies

We present a new inversion procedure applicable to bodies of revolution and cylindrical bodies. There are two significant improvements over the methods discussed in § 4. First, we only need to observe the scattered field for two closely spaced frequencies. Secondly, these frequencies can be high enough for spherical physical optics to be appropriate, provided that the shape of the scattering body is suitable (i.e. it is such that there is little multiple scattering). In fact, the higher these frequencies are the more accurately can details of body shape be recovered.

We find it convenient to introduce the notation

$$v_x = \partial v/\partial x, \tag{5.1}$$

where v is any scalar function and x is any variable.

(a) *Body of revolution*

We consider a body of revolution whose axis coincides with the polar axis of the spherical coordinates introduced in § 2. Using these coordinates we see that

$$r'_{\phi'} \equiv 0 \tag{5.2}$$

which implies that $E(\theta, \phi, k)$ is itself independent of ϕ, so that all available information is contained in $E(\theta, 0, k)$.

We consider values of k high enough for the integrals in (2.15) to be evaluated usefully by stationary phase. Because of (5.2), the integrals over ϕ' and θ' can be treated separately. It is convenient to deal with the former first. When $\phi = 0$, the phase of the integrand is stationary when $\phi' = 0$ and $\phi' = \pi$. Proceeding in the usual way (cf. Jones 1964, § 8.5) we find from (2.15) and (2.18) that

$$E(\theta, 0, k) \sim (-\mathrm{i}2k\pi/\sin\theta)^{\frac{1}{2}} \int_0^{\pi} (r'\sin\theta')^{\frac{1}{2}} \exp[\mathrm{i}2kr'\cos^2\{(\theta-\theta')/2\}]\,\mathrm{d}\theta'$$
$$+ (\mathrm{i}2k\pi/\sin\theta)^{\frac{1}{2}} \int_0^{\pi} (r'\sin\theta')^{\frac{1}{2}} \exp[\mathrm{i}2kr'\cos^2\{(\theta+\theta')/2\}]\,\mathrm{d}\theta'. \tag{5.3}$$

The phases of the two integrands in r.h.s. (5.3) are stationary when

$$\cos\{(\theta' \mp \theta)/2\} = 0 \tag{5.4}$$

and

$$\tan\{(\theta' \mp \theta)/2\} = r'_{\theta'}(\theta', 0)/r'(\theta', 0), \tag{5.5}$$

where the minus and plus signs apply to the first and second integrands respectively. Because the body is, by definition, symmetrical about the polar axis, we see that $r'_{\theta'} = 0$ when $\theta' = 0$ or $\theta' = \pi$.

Consequently, when $\theta = 0$ or $\theta = \pi$, both (5.4) and (5.5) give stationary phase points for both integrands at $\theta = 0$ and $\theta = \pi$. When $0 < \theta < \pi$, the only solution to (5.4) which lies within the range $[0, \pi]$ of the integrands in (5.3) is

$$\theta' = \pi - \theta \tag{5.6}$$

which applies only to the second integrand.

We cannot expect to obtain useful results from (5.3) when the surface of the body has sufficiently deep concavities that appreciable multiple scattering occurs, because (5.3) is based on physical optics which is not capable of predicting multiple scattering effects. Concavities in the body's surface are related to the occurrence of multiple stationary phase points in the integrands on r.h.s. (5.3). We must assume that each integrand possesses only one stationary phase point. The one for the first integrand is given by

$$\tan\{(\theta' - \theta)/2\} = r_{\theta'}(\theta', 0)/r'(\theta', 0) \tag{5.7}$$

which we assume has itself only one solution for $0 < \theta' < \pi$. The one for the second integrand is given by (5.6). We must assume that $|r'_{\theta'}(\theta', 0)|$ is never large enough that there is a solution to (5.5) for $0 < \theta' < \pi$, when the plus sign is taken. We can only obtain a recognizable reconstruction of the shape of the body when it is such that our assumptions are valid.

The recovery of $r'(\theta', 0)$ from (5.3) is very similar to the recovery of $\rho'(\phi')$ from the equivalent equation for a cylindrical body, which is discussed in § 5 (*b*) below. Since the illustrative examples which we present in § 6 concern cylindrical bodies, we feel that it is better to give the detailed analysis in the following sub-section.

(*b*) *Cylindrical body*

Stationary phase points of the integrand in (2.28) occur when

$$\cos\{(\phi' - \phi)/2\} = 0 \tag{5.8}$$

and

$$\tan\{(\phi' - \phi)/2\} = \rho'_{\phi'}(\phi')/\rho'(\phi'). \tag{5.9}$$

There is one solution to (5.8) for $0 \leqslant \phi' \leqslant 2\pi$:

$$\begin{aligned} \phi' &= \phi + \pi \quad (0 \leqslant \phi \leqslant \pi), \\ &= \phi - \pi \quad (\pi \leqslant \phi \leqslant 2\pi). \end{aligned} \tag{5.10}$$

For the same reasons as those we have previously given in the penultimate paragraph of § 5 (*a*), we must assume that there is only one solution to (5.9) for $0 \leqslant \phi' \leqslant 2\pi$. We say that

$$\phi' = \varphi \tag{5.11}$$

represents the solution to (5.9). We find it convenient to define

$$\rho = \rho'(\varphi); \quad \dot{\rho} = \rho'_{\phi'}(\varphi); \quad \ddot{\rho} = \rho'_{\phi'\phi'}(\varphi). \tag{5.12}$$

When (5.9) through (5.12) are invoked, the stationary phase approximation to (2.28) reduces to two integrals which correspond, respectively, to the first and second integrals on r.h.s. (5.3). The usual technique (cf. Jones 1964, § 8.5) gives

$$2E(\phi, k) - 1 \sim \frac{\exp(\mathrm{i}2k\rho)\cos^2\{(\varphi - \phi)/2\}}{\{\ddot{\rho}/\rho - 3(\dot{\rho}/\rho)^2 - 1\}^{\frac{1}{2}}\cos\{(\varphi - \phi)/2\}}. \tag{5.13}$$

Inspection of r.h.s. (5.13) reveals no obvious, direct way to recover ρ as a function of φ, and φ as a function of ϕ. However, the exponential is of modulus unity and, which is more important, it is the only factor on r.h.s. (5.13) that depends on k. This suggests that we should investigate the modulus of the partial derivative of $E(\phi, k)$ with respect to k. After some algrebraic manipulation we find from (5.9) and (5.11) through (5.13) the (approximate) formula

$$\rho[2E(\phi, k) - 1] = [1 + (\dot{\rho}/\rho)^2]\, E_k(\phi, k). \tag{5.14}$$

Suppose that we observe, or are given, $E(\phi, k)$ for two closely spaced wave numbers, $(k+\epsilon)$ and $(k-\epsilon)$ say. If ϵ is small enough, we can say that

$$E_k(\phi, k) \approx [E(\phi, k+\epsilon) - E(\phi, k-\epsilon)]/2\epsilon \tag{5.15}$$

and

$$E(\phi, k) \approx [E(\phi, k+\epsilon) + E(\phi, k-\epsilon)]/2 \tag{5.16}$$

to within some prescribed tolerance.

The formula (5.14) can be looked on as a differential equation for recovering $\rho = \rho'(\varphi)$ and $\varphi = \varphi(\phi)$. An initial condition is required to start the solution. We look for values of ϕ about which $E(\phi, k)$ is locally even, in the following sense. If ϕ_0 is such a value of ϕ then

$$[E(\phi_0 + \vartheta, k) - E(\phi_0 - \vartheta, k)]/E(\phi_0, k)$$

is smaller than some prescribed threshold over a range of ϑ. We denote the width (extent, length, support) of this range by R. We find the value of ϕ_0 for which R is greatest, and call it $\hat{\phi}_0$. We postulate that for the point $P' \in C$ whose angular coordinate is $\hat{\phi}_0$, the centre of curvature lies on the line OP, or on its extension. This is equivalent to assuming that $\rho'_{\phi'}(\hat{\phi}_0) = 0$, which when combined with (5.9), (5.11) and (5.12) gives

$$\varphi = \phi \quad \text{when} \quad \phi = \hat{\phi}_0. \tag{5.17}$$

This is sufficient to start a numerical solution to (5.14) for $\varphi = \varphi(\phi))$ and $\rho' = \rho'(\varphi)$. The latter describes the shape of the body, as the definitions (5.12) show.

6. Applications

We present examples of the reconstruction, by the inversion procedure described in § 5 (b), of the cross sections of the cylindrical bodies shown in figure 2. The scattered fields, on which the inversion procedure operates, were computed using our rigorous null field methods, themselves developed in (I).

In all examples we take

$$\epsilon = 0.005, \tag{6.1}$$

where ϵ is introduced in (5.15) and (5.16). For all the bodies shown in figure 2

$$\hat{\phi}_0 = 0 \tag{6.2}$$

where $\hat{\phi}_0$ is defined in the final paragraph of § 5 (b). The symmetries of all the bodies are such that one quarter of C completely defines the rest of it. Accordingly, we only show our reconstructed cross sections for ϕ' in the range $[0, \frac{1}{2}\pi]$. Note that this is equivalent to φ being restricted to the range $[0, \frac{1}{2}\pi]$, on account of the symmetries of the bodies and the definition (5.11) of φ in terms of ϕ'. We think it more graphic to relate our results to the wavelength λ of the field, rather than to its wave number k or its frequency $\omega/2\pi$. In terms of k, we write λ as

$$\lambda = 2\pi/k. \tag{6.3}$$

Since circular physical optics is exact for circular cylinders, we can reconstruct such cylinders perfectly. The greater the departure from circularity of the cross section of the body, the more difficult it is for us to reconstruct it accurately. Figure 3(*a*) shows that we can reconstruct elliptical cross sections of moderate ellipticity almost perfectly, even when the wavelength is only a little less than the smallest linear dimension of the body. Figure 3(*b*) confirms that the error in reconstructing the cross section tends to increase with the ellipticity.

The results presented in figure 4 illustrate two features of our (and any other, for that matter) reconstruction procedure. First, keeping constant the ratio of λ to the smallest linear dimension

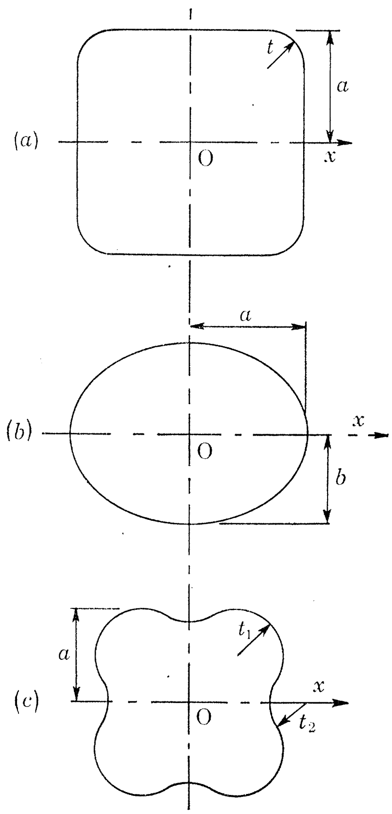

FIGURE 2. Cylindrical scattering bodies: (*a*) square cylinder with rounded corners; (*b*) elliptical cylinder; (*c*) cylinder with concavities.

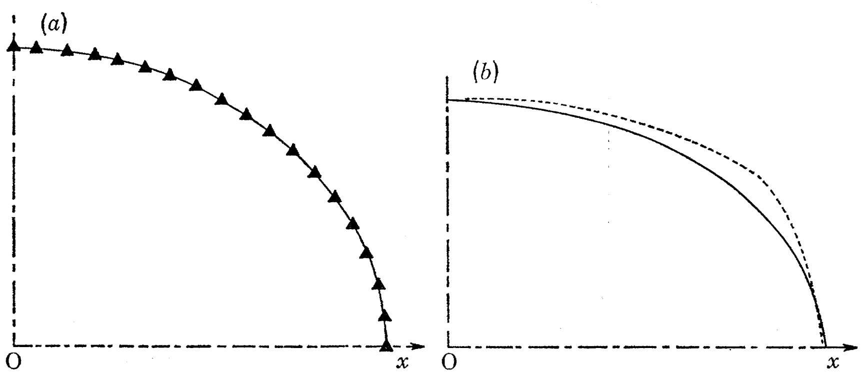

FIGURE 3. Reconstruction of the cross section of an elliptic cylinder (refer to figure 2*b*). (*a*) $b = 0.8a$; ——, boundary curve C; ▲, reconstructed points when $a = 1.5\lambda$ and $a = 2\lambda$. (*b*) $b = 0.65a$; ——, boundary curve C; – – –, reconstruction of C when $a = 2\lambda$.

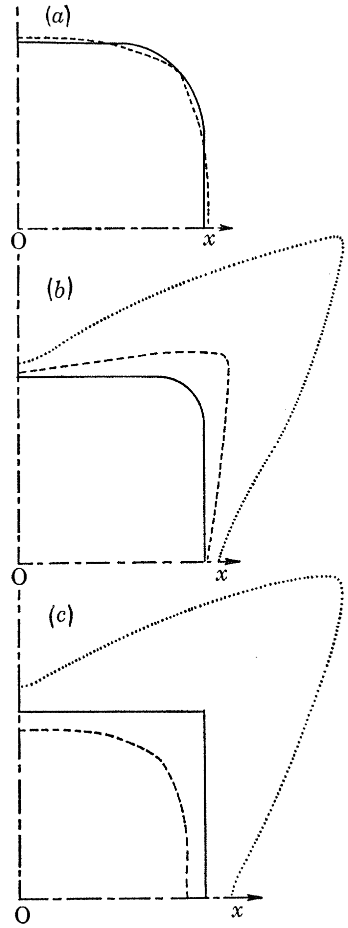

FIGURE 4. Reconstruction of the cross section of a square cylinder with rounded corners (refer to figure 2*a*). (*a*) $t = 0.5a$; ——, boundary curve C; – – –, reconstruction of C when $a = 2\lambda$. (*b*) $t = 0.25a$; ——, boundary curve C; ......, reconstruction of C when $a = 1.5\lambda$; – – –, reconstruction of C when $a = 2\lambda$. (*c*) $t = 0$; ——, boundary curve C; ......, reconstruction of C when $a = 1.5\lambda$; – – –, reconstruction of C when $a = 2\lambda$.

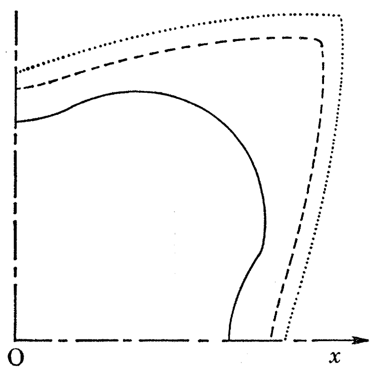

FIGURE 5. Reconstruction of the cross section of a cylinder with concavities (refer to figure 2*c*). ($t_1 = 0.5a$, $t_2 = 0.5a$); ——, boundary curve C; ......, reconstruction of C when $a = 2\lambda$; – – –, reconstruction of C when $a = 2.5\lambda$.

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of the body, the accuracy of reconstruction improves with the smoothness of the cross section – note that the differences between the dashed and full curves tend to decrease as we go from figure 4 (*c*) to figure 4 (*b*) to figure 4 (*a*). The second feature is that the error in reconstruction decreases with the ratio of λ to the smallest linear dimension – note the differences between the dotted, dashed and full curves in figures 4 (*b*) and (*c*).

A major problem with all shape reconstruction procedures, whether rigorously based or approximate, is to reproduce accurately concavities in scattering bodies. The reconstruction errors associated with the dashed curve in figure 5 are appreciably greater than those associated with the dashed curve in figure 4 (*c*), even though the wavelength is shorter for the former. Nevertheless, the reconstructions shown in figure 5 are encouraging and seem to be improving with decreasing wavelength. We found it inconvenient to investigate this trend in detail because of the large c.p.u. (central processing unit) times needed for calculating the scattered field accurately when $a/\lambda > 5$.

Given the scattered field, the c.p.u. time needed to compute each of the reconstructed cross sections shown in figures 3–5 was close to five seconds.

7. Conclusions

We suggest that in future research into analytical continuation techniques it may be worthwhile investigating and extending the test, developed in § 3 for determining numerically where a series expansion of a scattered field begins to diverge.

Theoretically, like previously reported methods, the method introduced in § 4 requires the scattered field to be known at all frequencies. Any attempts to introduce modifications designed to permit limited scattering data to be used must overcome the numerical instabilities noticed by Perry (1974). However, Young's (1976) results strongly suggest that this approach is worth pursuing further.

We feel that the inversion procedure which we present in § 5 and illustrate in § 6 is a significant improvement on previously reported techniques because it only requires that scattering data be available at two closely spaced frequencies, high enough for the wavelengths to be short enough to be compared with the linear dimensions of the scattering body. Even though our inversion procedure is based on the principle of stationary phase, and might therefore be expected only to work satisfactorily for very short wavelengths, the results presented in § 6 indicate that we can get useful results when the wavelength is comparable with the smallest linear dimension of the scattering body.

The results presented in figure 5 emphasize the importance of perfecting accelerated methods of solving direct scattering problems, e.g. the technique introduced in § 3 of (III). It would then be possible to compute economically the scattered fields from moderately large (in terms of the wavelength) bodies. This would allow a proper evaluation to be made of inversion methods, because it is such bodies which are usually of most physical and technical interest.

The formulae which we derive in § 5 are reminiscent of those reported by Keller (1959), (and later examined computationally by Weiss (1968)) who based his arguments on classical geometrical optics. Because we employ physical optics we are able to handle diffraction effects, which tend to introduce numerical instabilities when Keller's (1959) method is tried. It is perhaps worth remarking here that nobody has yet found a useful, general way of applying Keller's geometrical theory of diffraction to inverse scattering problems.

The methods based on planar physical optics (cf. Lewis 1969) can be adapted to permit estimates of body shape to be obtained by Fourier transformation of monochromatic scattered fields (cf. Bates 1969). However, the Fourier transformation relations only apply over limited ranges of aspect angles, which means that the scattered fields have to be partitioned into non-overlapping ranges of scattering angles. It is rarely clear how best to do this partitioning and it is always difficult to fit together the shapes obtained by Fourier transforming the individual parts of the field scattered from a particular body. All this emphasizes the convenience of the method we introduce in § 5, and it confirms that a worthwhile research problem is to extend the method so that it can be applied to asymmetrical non-cylindrical bodies.

We have found nothing useful to say concerning the uniqueness of our results. This is a mathematical shortcoming, but it is of little general scientific significance. The existing exact approaches to inverse scattering are so complicated and require such large quantities of difficult-to-measure data, to errors in which they are extremely sensitive, that it is difficult to believe they will ever be used in practice. The established approximate techniques are of very limited applicability. What are needed are computationally manageable improvements (such as we have introduced here) to known approximate methods. It is likely that uniqueness questions concerning these methods can only be answered heuristically, by computational experience.

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Appendix

It is virtually impossible to arrange physical sources such that (2.9) holds. However, it is possible to arrive at (2.9) by averaging over several incident fields.

A convenient point within the source distribution producing the incident field is chosen as a local origin, denoted by O_0. We place O_0 at a number, N say, of positions the nth position being denoted by O_{0n} all of which are at the same radial distance from the point O of figure 1. We, always maintain the same 'aspect' of the incident source distribution, in the sense that the line OO_0 can be thought of as a rigid rod glued into the incident source distribution, which is itself rigid. The rod OO_0 can be taken to possess a universal joint at O, thereby allowing O_0 to be moved to the points O_{0n}.

When O_0 is positioned at each of several of the O_{0n} we observe the number, N_n say, of scattered partial waves that are of significant amplitude. We denote by N' the largest of the N_n'. We then choose N such that

$$N = N'. \tag{A 1}$$

When O_0 is at O_{0n} we can write the incident field as $\psi_0 = \psi_0(r, \theta, \vartheta_n, \phi, \phi_n, k)$ where ϑ_n and ϕ_n are the angular coordinates of O_{0n}, in the spherical polar coordinate system (with origin O) introduced in § 2. The definitions introduced in this Appendix ensure that the error in the approximate relation

$$\frac{1}{N}\sum_{n=1}^{N} \psi_0(r, \theta, \vartheta_n, \phi, \phi_n, k) \approx \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} \psi_0(r, \theta, \phi, k) \sin(\theta)\, d\phi\, d\theta \tag{A 2}$$

is of the same order as the sum of the scattered partial waves whose amplitudes are considered too small to be significant. Inspection of (2.2) indicates that

$$\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} \psi_0(r,\theta,\phi,k)\sin(\theta)\,\mathrm{d}\phi\,\mathrm{d}\theta = -\mathrm{i}ka_{0,0}(k)\,j_0(kr) \qquad \text{(A 3)}$$

which is equivalent to (2.9).

When O_0 is at O_{0n} we can write the scattered field as $\psi = \psi(r,\theta,\vartheta_n,\phi,\phi_n,k)$. To the same level of approximation as before, we see that

$$\sum_{l=0}^{\infty}\sum_{j=-l}^{l} c_{j,l}\,b_{j,l}^{+}(k)\,h_l^{(2)}(kr)\,P_l^j(\cos\theta)\exp(\mathrm{i}j\phi) \approx \frac{1}{N}\sum_{n=1}^{N}\psi(r,\theta,\vartheta_n,\phi,\phi_n,k), \quad P\in\Upsilon_{++}, \qquad \text{(A 4)}$$

where the $b_{j,l}^{+}(k)$, of which only N have significant amplitude, characterize the scattered field when the incident field is characterized by (2.9).

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